On a class of generalized (κ, μ)-contact metric manifolds

By

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Abstract. We classify the 3-dimensional generalized (κ, μ)-contact metric manifolds, which satisfy the condition $\|grad\kappa\|$ =const.($\neq 0$). This class of manifolds is determined by two arbitrary functions of one variable.

1. Introduction. The tangent sphere bundle, of a Riemaniann manifold of constant sectional curvature admits a contact metric structure (η, ξ, ϕ, g) such that the characteristic vector field ξ belongs to the (κ, μ)-nullity distribution, for some real numbers κ and μ. This means that the curvature tensor R satisfies the condition

$$(*) \qquad R(X,Y)\xi = \kappa[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY]$$

for any vector fields X and Y, where h denotes, up to a scaling factor, the Lie derivative of the structure tensor field ϕ in the direction of ξ. The class of contact metric manifolds which satisfy $(*)$ has been classified in all dimensions, see [2],[3],[4].

On the other hand, the existence of 3-dimensional contact metric manifolds M satisfying $(*)$, with κ, μ non constant smooth functions on M, has been proved in [7], through the construction of examples. (In [7] it is also proved that for dimensions greater than 3 such manifolds do no exist). This class of Riemannian manifolds seems to be particulary large and we call such a manifold a generalized (κ, μ)-contact metric manifold (generalized (κ, μ)-c.m.m., in short).

In §3 of the present paper we give more examples of generalized (κ, μ)-c.m.m., with the additional property $\|grad\kappa\|$ =constant. Moreover, we remark that the condition $\|grad\kappa\|$ =constant, remains invariant under a D-homothetic deformation. Hence for any positive real number we can construct at least two such manifolds. The existence of these examples has been our motivation for their study.

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Initially, we prove that there exist two types of generalized (κ,μ)-c.m.m. with $\|grad\kappa\|$ =constant $\neq 0$. Type A, where $\mu = 2(1-\sqrt{1-\kappa})$ and type B, where $\mu = 2(1+\sqrt{1-\kappa})$. Next, in §4 we prove that such a manifold is covered by a global chart, in the coordinates of which we determine the functions κ and μ. In §5 we globally construct these manifolds. Finally, introducing a second transformation in §6, we succeed each member of this class is obtained by the first two examples given in §3, under such a transformation and a D-homothetic deformation.

All manifolds are assumed to be connected.

2. Preliminaries. In this section we collect some basic facts about contact metric manifolds. We refer to [1] for more detailed treatment. A differential $(2m+1)$ -dimensional manifold M is called a contact metric manifold if it carries a global differential 1-form η such that $\eta \wedge (d\eta)^m \neq 0$ everywhere. It is known that a contact manifold admits an almost contact metric structure (η,ξ,ϕ,g), i.e. a global vector field ξ, which will be called the characteristic vector field, a (1,1)-tensor field ϕ and a Riemannian metric g such that $\eta(\xi) = 1$, $\phi^2 = -Id + \eta \otimes \xi$ and $g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y)$ for all vector fields X, Y on M. Moreover (η,ξ,ϕ,g) can be chosen such that $d\eta(X,Y) = g(X,\phi Y)$. The manifold M together with the structure tensors (η,ξ,ϕ,g) is called a contact metric manifold and it is denoted by $M(\eta,\xi,\phi,g)$. Following [1], we define on M the (1,1)-tensor fields h and l by $h = \frac{1}{2}(\mathcal{L}_\xi\phi)$ and $l = R(.,\xi)\xi$, where $\mathcal{L}_\xi$ is the Lie differentiation in the direction of ξ and R the curvature tensor. The tensor fields h, l are self adjoint and satisfy $h\xi = l\xi = 0, Trh = 0, Tr\phi h = 0, h\phi + \phi h = 0$,

$$Trl = g(Q\xi,\xi), \tag{1}$$

where Q is the Ricci operator. Since h anti-commutes with ϕ, if X is an eigenvector of h corresponding to the eigenvalue λ, then ϕX is also an eigenvector of h corresponding to the eigenvalue $-\lambda$. If ∇ is the Riemannian connection of g, then $\nabla_\xi\phi = 0$,

$$\nabla_X\xi = -\phi X - \phi h X \quad (and\ so\ \nabla_\xi\xi = 0), \tag{2}$$

$$\nabla_\xi h = \phi - \phi l - \phi h^2. \tag{3}$$

Particularly, for the 3-dimensional case, the following formulas are valid ([6])

$$h^2 = (\frac{Trl}{2} - 1)\phi^2, \quad \frac{Trl}{2} \leq 1, \tag{4}$$

$$\sum_i (\nabla_{X_i} h) X_i = \phi Q\xi, \tag{5}$$

where $X_i, i = 1,2,3$, is an arbitrary orthonormal frame.

By a generalized (κ,μ)-contact metric manifold we mean a 3-dimensional contact metric manifold such that

$$R(X,Y)\xi = \kappa[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY], \quad X, Y \epsilon \mathcal{X}(\mathcal{M}), \tag{6}$$

where κ, μ are smooth functions on M, independent of the choice of vector fields X and Y.

The formulas in the next Lemma are known (see [5], [7]). For the sake of completeness we will give the outline of their proofs.

Lemma 1. On any generalized (κ, μ)-c.m.m. the following formulas are valid

$$h^2 = (\kappa - 1)\phi^2, \quad \kappa = \frac{Trl}{2} \leq 1 \tag{7}$$

$$\xi\kappa = 0, \quad hgrad\mu = grad\kappa. \tag{8}$$

Moreover, if $\kappa \neq 1$ everywhere on M, then

$$\nabla_X \xi = -(\lambda + 1)\phi X, \quad \nabla_{\phi X}\xi = (1 - \lambda)X, \tag{9}$$

$$\nabla_\xi X = -\frac{\mu}{2}\phi X, \quad \nabla_\xi \phi X = \frac{\mu}{2}X, \quad \nabla_X X = \frac{\phi X\lambda}{2\lambda}\phi X, \quad \nabla_{\phi X}\phi X = \frac{X\lambda}{2\lambda}X, \tag{10}$$

$$\nabla_{\phi X} X = -\frac{X\lambda}{2\lambda}\phi X + (\lambda - 1)\xi, \quad \nabla_X \phi X = -\frac{\phi X\lambda}{2\lambda}X + (\lambda + 1)\xi, \tag{11}$$

$$[\xi, X] = (1 + \lambda - \frac{\mu}{2})\phi X, \quad [\xi, \phi X] = (\lambda - 1 + \frac{\mu}{2})X, \tag{12}$$

$$[X, \phi X] = -\frac{\phi X\lambda}{2\lambda}X + \frac{X\lambda}{2\lambda}\phi X + 2\xi, \tag{13}$$

where $(\xi, X, \phi X)$ is a local orthonormal basis of eigenvectors of h, such that $hX = \lambda X, \quad \lambda = \sqrt{1 - \kappa} > 0$.

Proof. Using (6), we easily get $R(\xi, X)Y = \kappa[g(X, Y)\xi - \eta(Y)X] + \mu[g(hX, Y)\xi - \eta(Y)hX]$ and so by the definition of Q and (1) we get $Q\xi = 2\kappa\xi$ and $Trl = 2\kappa$. This and (4) imply (7). Using (6), $Q\xi = 2\kappa\xi$ and the well known formula

$$\begin{aligned} R(X, Y)Z &= g(Y, Z)QX - g(X, Z)QY + g(QY, Z)X \\ &\quad - g(QX, Z)Y - \frac{S}{2}(g(Y, Z)X - g(X, Z)Y) \end{aligned}$$

for $Y = Z = \xi$, we get

$$Q = aI + b\eta \otimes \xi + \mu h, \tag{14}$$

where S is the scalar curvature, $a = \frac{1}{2}(S - 2\kappa)$ and $b = \frac{1}{2}(6\kappa - S)$. Using (6), $\phi^2 = -Id + \eta \otimes \xi$ and the definition of l we find $l = -\kappa\phi^2 + \mu h$. This, together with (3) , (7) and $h\phi + \phi h = 0$ give $\nabla_\xi h = \mu h\phi$. Differentiating $h^2 = (\kappa - 1)\phi^2$ with respect to ξ and using $\nabla_\xi \phi = 0$ and the last equation we get the first equation of (8). Differentiating (14) with respect to an orthonormal basis $X_i, i = 1, 2, 3$, and using (2), $Trh\phi = 0$, $\phi\xi = h\xi = 0$, $Q\xi = 2\kappa\xi$ and (5) we find

$$\sum_i (\nabla_{X_i} Q)X_i = grada + (\xi b)\xi + hgrad\mu.$$

Comparing this with the well known formula $\sum_i (\nabla_{X_i} Q)X_i = \frac{1}{2} gradS$, we get $hgrad\mu = grad\kappa$. Relations (9) are immediate consequences of (2). The first two relations of (10) are obtained from (6), for $Y = \xi$, and the definition of the curvature tensor. Using (5) we get the last two relations of (10). Relations (11) follow from (9) and (10), while (12) and (13) are immediate consequences of (9)-(11). We denote that the existence of the local basis $(\xi, X, \phi X)$ is proved in [7].

3. Examples. 1. (Type A). We consider the 3-dimensional manifold $M = \{(x,y,z) \epsilon R^3 / z < 1\}$, where (x,y,z) are the standard coordinates in R^3. The 1-form $\eta = dx + 2ydz$ defines a contact structure on M with characteristic vector field $\xi = \frac{\partial}{\partial x}$. Let g, ϕ be the Riemannian metric and the (1,1)-tensor field given by

$$g = \begin{pmatrix} 1 & 0 & -a \\ 0 & 1 & -b \\ -a & -b & 1+a^2+b^2 \end{pmatrix}, \quad \phi = \begin{pmatrix} 0 & -a & ab \\ 0 & -b & 1+b^2 \\ 0 & -1 & b \end{pmatrix}$$

with respect to the basis $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$, where $a = -2y$ and $b = 2x\sqrt{1-z} + \frac{y}{4(1-z)}$. The tensor fields (η, ξ, ϕ, g) define a generalized (κ, μ)-contact metric manifold with $\kappa = z$ (and so $\|grad\kappa\| = 1$) and $\mu = 2(1 - \sqrt{1-z})$.

2. (Type B). On the manifold M of the previous example we define the tensor fields (η, ξ, ϕ, g) by $\eta = dx - 2ydz$, $\xi = \frac{\partial}{\partial x}$,

$$g = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & -b \\ a & -b & 1+a^2+b^2 \end{pmatrix}, \quad \phi = \begin{pmatrix} 0 & -a & ab \\ 0 & b & -1-b^2 \\ 0 & 1 & -b \end{pmatrix}.$$

Then $M(\eta, \xi, \phi, g)$ is a generalized (κ, μ)-contact metric manifold with $\kappa = z$ and $\mu = 2(1 + \sqrt{1-z})$.

3. Let $M(\eta, \xi, \phi, g)$ be a contact metric manifold. By a D_a-homothetic deformation (see [8],[2]) we mean a change of structure tensors of the form

$$\bar{\eta} = a\eta, \quad \bar{\xi} = \frac{1}{a}\xi, \quad \bar{\phi} = \phi, \quad \bar{g} = ag + a(a-1)\eta \otimes \eta,$$

where a is a positive number.The curvature tensor R and the tensor h transform in the following manner [2]: $\bar{h} = \frac{1}{a}h$ and $a\bar{R}(X,Y)\bar{\xi} = R(X,Y)\xi + (a-1)^2[\eta(Y)X - \eta(X)Y] - (a-1)[(\nabla_X \phi)Y - (\nabla_Y \phi)X + \eta(X)(Y + hY) - \eta(Y)(X + hX)]$ for any X, Y. Additionally, it is well known [9, pp 446,447], that any 3-dimensional contact metric manifold satisfies $(\nabla_X \phi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX)$. Using these we have that if $M(\eta, \xi, \phi, g)$ is a generalized (κ, μ)-c.m.m., then $M(\bar{\eta}, \bar{\xi}, \bar{\phi}, \bar{g})$ is also a generalized $(\bar{\kappa}, \bar{\mu})$-c.m.m. with $\bar{\kappa} = \frac{\kappa + a^2 - 1}{a^2}$ and $\bar{\mu} = \frac{\mu + 2(a-1)}{a}$ ([7]). Therefore, if $M(\eta, \xi, \phi, g)$ satisfies $\|grad\kappa\|_g = d$ (const.), then $M(\bar{\eta}, \bar{\xi}, \bar{\phi}, \bar{g})$ satisfies $\|grad\bar{\kappa}\|_{\bar{g}} = da^{-\frac{5}{2}}$. It follows from the fact that, if $(\xi, X, \phi X)$ is an orthonormal basis with respect to g, then $(\frac{1}{a}\xi, \frac{1}{\sqrt{a}}X, \frac{1}{\sqrt{a}}\phi X)$ is an orthonormal basis with respect to $\bar{g}$.

As a result of the above and examples 1,2, we have the following Proposition.

Proposition 2. For any positive number, there exist at least two generalized (κ, μ)-c.m.m. with $||grad\kappa||$ =constant $\neq 0$.

Remark 1. (i) Using the fact that, any generalized (κ, μ)-c.m.m. with $||grad\kappa||_g = d \neq 0$ (const.) is D_a-deformed in another generalized $(\bar{\kappa}, \bar{\mu})$-c.m.m. with $||grad\bar{\kappa}||_{\bar{g}} = da^{-\frac{5}{2}}$, for any positive a and choosing $a = d^{\frac{2}{5}}$, it is enough to study those generalized (κ, μ)-c.m.m. with $||grad\kappa|| = 1$.
(ii) If $d = 0$, then κ is constant. Therefore, if $\kappa = 1$, then M is a Sasakian manifold [2], while for $\kappa \neq 1$, μ =constant [7].
(iii) A D_a-homothetic deformation preserves the type of a generalized (κ, μ)-contact metric manifold with $||grad\kappa||$ =const.

4. Main results. From now on, we suppose that $M(\eta, \xi, \phi, g)$ is a generalized (κ, μ)-contact metric manifold with $||grad\kappa|| = 1$. Because $hgrad\mu = grad\kappa$, we have $h \neq 0$ and so $\kappa \neq 1$ everywhere on M as it follows from (7). We denote by $(\xi, X, \phi X)$ a local orthonormal frame of eigenvectors of h such that $hX = \lambda X$, $\lambda = \sqrt{1-\kappa} > 0$. The next Lemma inform us that there exist 2-types of such manifolds.

Lemma 3. Let M be a generalized (κ, μ)-c.m.m. with $||grad\kappa|| = 1$. Then $\mu = 2(1-\lambda)$ or $\mu = 2(1+\lambda)$. In the first case (type A), the following are valid,
$X\kappa = 1, \phi X\kappa = 0, [\xi, X] = 2\lambda\phi X, [\xi, \phi X] = 0$ and $[X, \phi X] = -\frac{1}{4\lambda^2}\phi X + 2\xi$.
In the second case (type B), the following are valid,
$\phi X\kappa = 1, X\kappa = 0, [\xi, X] = 0, [\xi, \phi X] = 2\lambda X$ and $[X, \phi X] = \frac{1}{4\lambda^2}X + 2\xi$.

Proof. Using $\xi\kappa = 0$ and $||grad\kappa|| = 1$ we have

$$grad\kappa = (X\kappa)X + (\phi X\kappa)\phi X \tag{15}$$

and

$$(X\kappa)^2 + (\phi X\kappa)^2 = 1. \tag{16}$$

Differentiating (16) with respect to ξ and using (8) and (12) we get successively

$$(\xi X\kappa)(X\kappa) + (\xi\phi X\kappa)(\phi X\kappa) = 0$$
$$([\xi, X]\kappa)(X\kappa) + ([\xi, \phi X]\kappa)(\phi X\kappa) = 0$$
$$\lambda(X\kappa)(\phi X\kappa) = 0$$

and since $\lambda \neq 0$,

$$(X\kappa)(\phi X\kappa) = 0. \tag{17}$$

We consider the open sets

$$A = \{P\epsilon M/(X\kappa)(P) \neq 0\} \quad \text{and} \quad B = \{P\epsilon M/(\phi X\kappa)(P) \neq 0\}.$$

Because $||grad\kappa|| \neq 0$, we have $A \cap B = \emptyset$ and $A \cup B = M$. Moreover, by the connectness of M we get $A = M$ and $B = \emptyset$ or $B = M$ and $A = \emptyset$. We distinguish two cases.
Case 1. Let $A = M$. Then, (17) gives $\phi X\kappa = 0$. Using this, $X\kappa \neq 0$, $\xi\kappa = 0$, then the second of (12) gives $\mu = 2(1-\lambda)$ and $[\xi, \phi X] = 0$. Moreover, from (16) we have $X\kappa = \pm 1$. Without loss of generality we may assume that $X\kappa = 1$, differently we choose the basis $(\xi, -X, -\phi X)$. Differentiating $\lambda^2 = 1 - \kappa$ and using $X\kappa = 1$ and $\phi X\kappa = 0$ we get $X\lambda = -\frac{1}{2\lambda}$ and $\phi X\lambda = 0$. Substituting these in (12) and (13) we have $[\xi, X] = 2\lambda\phi X$ and $[X, \phi X] = -\frac{1}{4\lambda^2}\phi X + 2\xi$.
Case 2. Let $B = M$. Then $\phi X\kappa \neq 0$ and $X\kappa = 0$. Working as in case 1 we finally get $\mu = 2(1 + \lambda)$, $\phi X\kappa = 1$, $[\xi, \phi X] = 2\lambda X$ and $[X, \phi X] = \frac{1}{4\lambda^2}X + 2\xi$. This completes the proof of the Lemma.

Remark 2. In the case of type A ($\mu = 2(1-\lambda)$), we have $X = grad\kappa$ and in the case of type B ($\mu = 2(1+\lambda)$), we have $X = -\phi grad\kappa$, as they follow from (15). Because the function κ is globally defined on M, we conclude that the orthonormal frame $(\xi, X, \phi X)$ of eingenvectors of h is globally defined on M.

Remark 2, leads us to the following Proposition.

Proposition 4. Any generalized (κ, μ)-c.m.m. with $||grad\kappa||$ =const.$\neq 0$ is parallelizable.

In the next Lemma, we construct a suitable chart, whose domain is the whole of the manifold.

Lemma 5. Let M be a generalized (κ, μ)-c.m.m. with $||grad\kappa|| = 1$. Then, there exists a chart (x, y, z) whose domain covers M. Moreover,$\kappa = z$, $z < 1$, everywhere on M.

Proof. According to Lemma 3 we distinguish two cases.
Case 1. Let $\mu = 2(1 - \lambda)$. Because $[\xi, \phi X] = 0$, the distribution which is obtained by ϕX and ξ is integrable. So for any point $P \epsilon M$ there exists a chart $\{V, (\bar{x}, \bar{y}, \bar{z})\}$ at P, such that

$$\xi = \frac{\partial}{\partial \bar{x}}, \phi X = \frac{\partial}{\partial \bar{y}} \quad \text{and} \quad X = a\frac{\partial}{\partial \bar{x}} + b\frac{\partial}{\partial \bar{y}} + c\frac{\partial}{\partial \bar{z}},$$

where $a, b, c, (c \neq 0)$, are smooth functions on V. Now, we consider on V the linearly independent vector fields $\xi, \phi X, W = c\frac{\partial}{\partial \bar{z}}$. An easy calculation implies $\frac{\partial c}{\partial y} = 0, \frac{\partial c}{\partial x} = 0$ and so $[\phi X, W] = [\xi, W] = [\xi, \phi X] = 0$. This means that there exists a chart $\{U, (x, y, z')\}$ at P such that $\xi = \frac{\partial}{\partial x}, \phi X = \frac{\partial}{\partial y}, W = \frac{\partial}{\partial z'}$. On U we have $\xi = \frac{\partial}{\partial \bar{x}} = \frac{\partial}{\partial x}, \phi X = \frac{\partial}{\partial \bar{y}} = \frac{\partial}{\partial y}$ and $X = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + \frac{\partial}{\partial z'}$, as it follows from $W = \frac{\partial}{\partial z'} = c\frac{\partial}{\partial \bar{z}} = X - a\frac{\partial}{\partial x} - b\frac{\partial}{\partial y}$. Using it, $X\kappa = 1, \frac{\partial \kappa}{\partial y} = \phi X\kappa = 0$ and $\frac{\partial \kappa}{\partial x} = \xi\kappa = 0$ we get $\frac{\partial \kappa}{\partial z'} = 1$ and so $\kappa = z' + d$, where d is an integration constant. The substitution $z = z' + d$, locally completes the proof of the Lemma in case 1.

Case 2. Let $\mu = 2(1+\lambda)$. Working, as in case 1, for the integrable distribution, which is obtained by ξ and X ($[\xi, X] = 0$), we finally find that there exists a chart (x,y,z) at $P\epsilon M$ on whose domain U, $\kappa = z, \xi = \frac{\partial}{\partial x}, X = \frac{\partial}{\partial y}$ and $\phi X = a'\frac{\partial}{\partial x} + b'\frac{\partial}{\partial y} + \frac{\partial}{\partial z}$, where a', b' are smooth functions on M. Since $\lambda = \sqrt{1-\kappa}$, it is obvious that $z < 1$ in both cases.

Now, we will prove that domain U of the above chart can be extended such as to be the whole of M. We will prove case 1, as far as the proof of case 2 is analogous. We suppose that (A, ψ) is a chart at P such that the open set A is the largest possible extension of U. Let $A \neq M$. Then, for any $q\epsilon\partial A$, there exists (as we have proved) a chart $(V, \bar{\psi}(\bar{x}, \bar{y}, \bar{z}))$ at q, such that $\kappa = \bar{z}, \xi = \frac{\partial}{\partial \bar{x}}, \phi X = \frac{\partial}{\partial \bar{y}}$. On $A \cap V$ we have $\bar{z} = z, \xi = \frac{\partial}{\partial \bar{x}} = \frac{\partial}{\partial x}$ and $\phi X = \frac{\partial}{\partial \bar{y}} = \frac{\partial}{\partial y}$. From these, we get $(\bar{x}, \bar{y}, \bar{z}) = (x + c_1, y + c_2, z)$, where c_1, c_2 are integration constants. We consider the smooth function ω on $A \cup V$, such that $\omega = \psi$ on A and $\omega = \bar{\psi} - (c_1, c_2, 0)$ on V. Then, $(A \cup V, \omega)$ defines a new chart of M at P, whose domain $A \cup V \supset A$. By this contradiction, we conclude that $A = M$. This completes the proof of the Lemma.

An immediate and expected consequence of the above result is the following Corollary.

Corollary 6. There are no compact, generalized (κ, μ)-contact metric manifolds with $\|grad\kappa\| = \text{const.} \neq 0$.

Now, we will state and prove our main result.

Theorem 7. Let $M(\eta, \xi, \phi, g)$ be a 3-dimensional generalized (κ, μ)-contact metric manifold with $\|grad\kappa\| = 1$. Then M is covered by a chart $(x,y,z), z < 1$, such that $\kappa = z$ and $\mu = 2(1 - \sqrt{1-z})$ or $\mu = 2(1+\sqrt{1-z})$. In the first case ($\mu = 2(1-\sqrt{1-z})$), the following are valid,

$$\xi = \frac{\partial}{\partial x}, \quad \phi X = \frac{\partial}{\partial y} \quad \text{and} \quad X = a\frac{\partial}{\partial x.} + b\frac{\partial}{\partial y} + \frac{\partial}{\partial z}.$$

In the second case ($\mu = 2(1+\sqrt{1-z})$), the following are valid,

$$\xi = \frac{\partial}{\partial x}, \quad X = \frac{\partial}{\partial y} \quad \text{and} \quad \phi X = a'\frac{\partial}{\partial x} + b'\frac{\partial}{\partial y} + \frac{\partial}{\partial z},$$

where $a(x,y,z) = -2y + f(z)$, $a'(x,y,z) = 2y + h(z)$ $b(x,y,z) = b'(x,y,z) = 2x\sqrt{1-z} + \frac{y}{4(1-z)} + r(z)$ and f, r, h are smooth functions of z.

Proof. Because of Lemma 5 (see, also its proof) we just have to calculate functions a, b, a', b'.
Let $\mu = 2(1 - \sqrt{1-z})$. Then

$$[\xi, X] = \frac{\partial a}{\partial x}\frac{\partial}{\partial x} + \frac{\partial b}{\partial x}\frac{\partial}{\partial y} \quad \text{and} \quad [X, \phi X] = -\frac{\partial a}{\partial y}\frac{\partial}{\partial x} - \frac{\partial b}{\partial y}\frac{\partial}{\partial y}.$$

Combining these, with $[\xi, X] = 2\lambda\phi X = 2\lambda\frac{\partial}{\partial y}$ and $[X, \phi X] = -\frac{1}{4\lambda^2}\phi X + 2\xi = -\frac{1}{4\lambda^2}\frac{\partial}{\partial y} + 2\frac{\partial}{\partial x}$ (see, Lemma 3), we get

$$\frac{\partial a}{\partial x} = 0, \quad \frac{\partial b}{\partial x} = 2\lambda, \quad \frac{\partial a}{\partial y} = -2 \quad \text{and} \quad \frac{\partial b}{\partial y} = \frac{1}{4\lambda^2}.$$

It follows from this system that, $a = -2y + f(z)$ and $b = 2x\sqrt{1-z} + \frac{y}{4(1-z)} + r(z)$, where $f(z), r(z)$ are integration functions.
Now, let $\mu = 2(1 + \sqrt{1-z})$. We have

$$[\xi, \phi X] = \frac{\partial a'}{\partial x}\frac{\partial}{\partial x} + \frac{\partial b'}{\partial x}\frac{\partial}{\partial y} \quad \text{and} \quad [X, \phi X] = \frac{\partial a'}{\partial y}\frac{\partial}{\partial x} + \frac{\partial b'}{\partial y}\frac{\partial}{\partial y}.$$

Combining these, with $[\xi, \phi X] = 2\lambda X = 2\lambda\frac{\partial}{\partial y}$ and $[X, \phi X] = \frac{1}{4\lambda^2}X + 2\xi = \frac{1}{4\lambda^2}\frac{\partial}{\partial y} + 2\frac{\partial}{\partial x}$ we get

$$\frac{\partial a'}{\partial x} = 0, \quad \frac{\partial b'}{\partial x} = 2\lambda, \quad \frac{\partial a'}{\partial y} = 2 \quad \text{and} \quad \frac{\partial b'}{\partial y} = \frac{1}{4\lambda^2}$$

and so $a'(x, y, z) = 2y + h(z)$ and $b'(x, y, z) = 2x\sqrt{1-z} + \frac{y}{4(1-z)} + r(z)$, where $h(z), p(z)$ are integration functions. This competes the proof of the Theorem.

Remark 3. The functions a, b, a', b' of Theorem 7 determine the manifold completely, as we will see in the next paragraph. There, using the conclusion of Lemma 5, we will construct in R^3 all the generalized (κ, μ)-c.m.m. with $\|grad\kappa\| = 1$.

5. Construction. Let $M = \{(x, y, z)\epsilon R^3 / z < 1\}$ and $f, r : M \to R$ be arbitrary functions of z. We consider the linearly independent vector fields

$$e_1 = \frac{\partial}{\partial x}, \quad e_2 = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + \frac{\partial}{\partial z}, \quad e_3 = \frac{\partial}{\partial y},$$

where $a(x, y, z) = -2y + f(z), b(x, y, z) = 2x\sqrt{1-z} + \frac{y}{4(1-z)} + r(z)$. Let g be the Riemannian metric defined by $g(e_i, e_j) = \delta_{ij}, (i, j = 1, 2, 3)$, ∇ the Riemannian connection and R the curvature tensor of g. Putting $\lambda = \sqrt{1-z}$, we easily get $[e_1, e_3] = 0, [e_1, e_2] = 2\lambda e_3, [e_2, e_3] = -\frac{1}{4\lambda^2}e_3 + 2e_1$. Moreover, we define the 1-form η and the (1,1)-tensor field ϕ by $\eta(.) = g(., e_1)$ and $\phi e_1 = 0, \phi e_2 = e_3, \phi e_3 = -e_2$. Because $\eta \wedge d\eta \neq 0$ everywhere on M, η is a contact form. Using the linearity of ϕ, $d\eta$ and g we find $\eta(e_1) = 1, \phi^2 Z = -Z + \eta(Z)e_1, d\eta(Z, W) = g(Z, \phi W)$ and $g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W)$ for any $Z, W\epsilon\mathcal{X}(\mathcal{M})$. Hence $M(\eta, e_1, \phi, g)$ defines a contact metric structure on M. Putting $\xi = e_1, X = e_2, \phi X = e_3$ and using the well known formula

$$2g(\nabla_Y Z, W) = Yg(Z, W) + Zg(W, Y) - Wg(Y, Z) \\ -g(Y, [Z, W]) - g(Z, [Y, W]) + g(W, [Y, Z]),$$

we find the formulas (9)-(13). Moreover, for the tensor field h we get $h\xi = 0, hX = \lambda X, h\phi X = -\lambda\phi X$. Using the above relations and the definition of the curvature tensor, we finally get that $M(\eta, \xi, \phi, g)$ is a generalized (κ, μ)-c.m.m. (of type A) with $\kappa = z$ and $\mu = 2(1 - \sqrt{1-z})$.

In order to construct an arbitrary generalized (κ, μ)-c.m.m. with $\|grad\kappa\| = 1$ of type B, we work analogously on the same manifold M, considering the vector fields

$$e_1 = \frac{\partial}{\partial x}, \quad e_2 = \frac{\partial}{\partial y} \quad e_3 = a'\frac{\partial}{\partial x} + b'\frac{\partial}{\partial y} + \frac{\partial}{\partial z},$$

where $a' = 2y + f(z)$, $\quad b' = 2x\sqrt{1-z} + \frac{y}{4(1-z)} + r(z)$. The tensor fields g, η, ϕ are defined by $g(e_i, e_j) = \delta_{ij}, (i, j = 1, 2, 3)$, $\eta(.) = g(., e_1)$, $\phi e_1 = 0, \phi e_2 = e_3$ and $\phi e_3 = -e_2$. Putting $\xi = e_1, X = e_2$ and $\phi X = e_3$ we finally find that $M(\eta, \xi, \phi, g)$ is a generalized (κ, μ)-c.m.m. (of type B), with $\kappa = z$ and $\mu = 2(1 + \sqrt{1-z})$.

Remark 4. The examples 1 and 2 of §3 correspond in the special case $f = 0, r = 0$.

In §3 we have seen that a D_a-homothetic deformation tranforms a generalized (κ, μ)-c.m.m. with $\|grad\kappa\| = 1$ to another generalized $(\bar{\kappa}, \bar{\mu})$-c.m.m. with $\|grad\bar{\kappa}\| = d \neq 1$(const.). In the next paragraph we will introduce a second transformation, which transforms a generalized (κ, μ)-c.m.m. $M(\eta, \xi, \phi, g)$ with $\|grad\kappa\| = 1$ to another generalized (κ, μ)-c.m.m. $M(\bar{\eta}, \bar{\xi}, \bar{\phi}, \bar{g})$ with the same κ, μ, $\|grad\kappa\|_{\bar{g}} = 1$ and of the same type.

6. Another transformation. Let $M(\eta, \xi, \phi, g)$ be a generalized (κ, μ)-c.m.m. with $\|grad\kappa\| = 1$, and f, r smooth functions on M such that $\xi f = \xi r = 0$ and $(\phi grad\kappa)f = (\phi grad\kappa)r = 0$. We consider the vector fields

$$\bar{\xi} = \xi, \quad \bar{X} = grad\kappa + f\xi + r(\phi grad\kappa), \quad \bar{Y} = \phi grad\kappa$$

and we define the tensor fields $\bar{g}, \bar{\eta}, \bar{\phi}$ as follows,

$$\bar{g}(\bar{\xi}, \bar{\xi}) = \bar{g}(\bar{X}, \bar{X}) = \bar{g}(\bar{Y}, \bar{Y}) = 1, \bar{g}(\bar{\xi}, \bar{X}) = \bar{g}(\bar{\xi}, \bar{Y}) = \bar{g}(\bar{X}, \bar{Y}) = 0$$
$$\bar{\eta}(.) = \bar{g}(., \bar{\xi}), \quad \bar{\phi}\bar{\xi} = 0, \bar{\phi}\bar{X} = \bar{Y}, \bar{\phi}\bar{Y} = -\bar{X}.$$

Then $M(\bar{\eta}, \bar{\xi}, \bar{\phi}, \bar{g})$ is a generalized (κ, μ)-c.m.m. with $\|grad\kappa\|_{\bar{g}} = 1$ (with the same κ, μ).

To prove it, we distinguish two cases: $\mu = 2(1-\lambda)$ and $\mu = 2(1+\lambda)$. We will prove the first case, because the proof of the second case is similar. Let $\mu = 2(1-\lambda)$. Then, as we have seen in Lemma 3, $\xi\kappa = 0, (grad\kappa)\kappa = 1, (\phi grad\kappa)\kappa = 0$. Therefore, there exists a global coordinate system (x, y, z), (see, Theorem 7 and its proof) such that , $\kappa = z, \xi = \frac{\partial}{\partial x}, \phi grad\kappa = \frac{\partial}{\partial y}$ and $grad\kappa = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + \frac{\partial}{\partial z}$, where $a = -2y + h(z), b = 2x\sqrt{1-z} + \frac{y}{4(1-z)} + \nu(z)$ and h, ν smooth functions of z. Then $\bar{X} = (-2y + F(z))\frac{\partial}{\partial x} + (2x\sqrt{1-z} + \frac{y}{4(1-z)} + G(z))\frac{\partial}{\partial y} + \frac{\partial}{\partial z}$ and $\bar{Y} = \frac{\partial}{\partial y}$, where $F = f + h$ and $G = r + \nu$. According to the construction of §5, $M(\bar{\eta}, \bar{\xi}, \bar{\phi}, \bar{g})$ is also a generalized (κ, μ)-c.m.m. with $\kappa = z$ and $\mu = 2(1 - \sqrt{1 - z})$.

Remark 5. Any generalized (κ, μ)-c.m.m. with $\|grad\kappa\|$ =const. $\neq 0$ can be obtained by examples 1 and 2 of §3, under the above transformation and a D_a-homothetic deformation.

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